THE DEFINITION OF NUMBER.

"SCIENCE," writes Gaston Milhaud,[1] "in enouncing its ever-increasing series of truths, obviously supplies—whether one reflect upon it or not—the most powerful argument against scepticism. And in this respect mathematics plays a special rôle by reason of the evidence which clothes all its propositions and by reason of the complete satisfaction which its demonstrations give to our thirst for comprehension. There, at least, is a domain where thought in search of clarity, of evidence, and of light, exercises itself in an ideal fashion. Everywhere else, discussion is founded on the right to proclaim as certain an enounced truth, and accord upon the value and legitimacy of each insight comes but slowly: in mathematics this is not so. If, for the choice of axioms, we give ourselves voluntarily to philosophical investigations whose conclusions vary, at bottom there is no one ready to abandon the postulates of ancient geometry, and the question was not even proposed by the Greeks. As to demonstrations, it seems impossible that two minds, however different they may be—granting their disposal, at need, of obvious misunderstandings—will not speedily agree upon the rigor of the reasoning, and consequently upon the rigor of the conclusions. And whether one is aware of it or not, the habit of such a movement of thought creates in us a naive confidence in the puissance of our under-

[1] *Les Philosophes-géomètres de la Grèce,* 1900; pp. 2-3.

standing,—so that it would be a miracle if the philosophical geometer did not somewhere testify to it, did not sometimes under the most penetrating conceptions bear along a disconcerting dogmatism."

We may take this statement, I think, as a fair representation at once of the fascination and the dangers which beset mathematical reasonings. There is no field of human thought, I imagine, which yields so paradoxical a feeling of freedom and of constraint as does mathematics: the freedom springing from the twofold consciousness, first, of our having chosen the postulates from which we proceed, and second, of the endlessness of the possible elaborations of our reasonings; the constraint arising from our sense of the undeniableness, and therefore the necessity, of mathematical demonstrations,—i. e., from their freedom from contradiction. Thus from mathematics we derive the satisfaction which our instinct for law and order always yields in finding itself fulfilled, without at the same time sacrificing our self-gratifying conviction of the importance of the human factor in the operations performed. In the study of physical nature there is always a certain abasement of humanity, due to the passive attitude of scientific observation, accompanied by a feeling of outer and brute constraint; but the mathematician, with an even greater assurance of the necessity of his results, bears with him also a lively consciousness of the significance of his own activity in bringing about these results, and so attains, as it were, a kind of Zeus-like supremacy to the fated ends which, while they bind him, are yet his own enactment.

But is not this doubly reason for caution against mathematical dogmatism? and especially that form of it which rests its denial of our more ordinary intuitions, not upon its eventual translatability, but upon its untranslatability into the forms of our common human experience? Doubtless truth is difficult and obscure; but dare we concede that

it is so ineffably obscure as to transcend the discourse of life? Of course I am speaking of the modern science of logistic.[2]

I.

What is the meaning of number? and in what sense are the hairs of our heads and the other phenomena of nature numbered? This is the question.

The old-fashioned view of number found its essence to lie in *discontinuity* coupled with a notion of *series*. "Number is discontinuous," says Clerk-Maxwell;[3] "we pass from one number to the next *per saltum*." The perception of the discontinuity was regarded as empirical and intuitive. In the language of Aristotle, "We perceive number by the negation of continuity, and also by the special senses, for each sensation is a unity."[4] The perception of the series was usually accredited to the act of counting, though this was often also somewhat confusedly regarded as an act of adding. If I speak of this view in a past tense, it is only because of its long history; not that it is dead.

In the thinking of such men as Hobbes and Locke this conception eventuates in an out-and-out nominalism. "Number," quoth Hobbes,[5] "is exposed either by the exposition of points or of the names of number, *one, two, three, etc.*; and those points must not be contiguous, so as that they cannot be distinguished by notes, but they must be so placed that they may be *discerned* one from another; for from this it is that number is called *discrete quantity*, whereas all quantity which is designed by motion is called *continual quantity*. But that number may be exposed by the names of number it is necessary that they be recited

[2] Which is, by the way, a somewhat unhappy name; for with the Greeks "arithmetic" was the more, "logistic" the less theoretic science.

[3] *Encyc. Brit.*, 9th ed., III, 37.

[4] *De Anima*, 425*a*, 5.

[5] *Concerning Body*, XII, 5.

by heart and in order, as one, two, three, etc.; for by saying one, one, one, and so forward, we know not what number we are at beyond two or three; which also appear to us in this manner not as number, but as figure."

It is always worth while citing Locke in connections of this kind, not because of the analytical value of his expositions, which is usually slight, but because he gives, with a dogmatic perspicuousness that leaves nothing to be desired, the first reflections of ordinary common sense. He says:[6] "By the repeating the idea of an unit and joining it to another unit, we make thereof one collective idea marked by the name two: and whosoever can do this, and proceed on, still adding one more to the last collective idea which he had of any number and gave a name to it, may count, or have ideas for several collections of units distinguished from one another, as far as he hath a series of names for following numbers, and a memory to retain that series with their several names; all numeration being but still the adding of one unit more, and giving to the whole together, as comprehended in one idea, a new or distinct name or sign, whereby to know it from those before and after, and distinguish it from every smaller or greater multitude of units. So that he that can add one to one, and so to two, and so go on with his tale, taking still with him the distinct names belonging to every progression; and so again, by subtracting an unit from each collection, retreat and lessen them; is capable of all the ideas of number within the compass of his language, or for which he hath names, though perhaps not of more."

In this account it is obvious that Locke presupposes: (*a*) the notion of *unity*, which, indeed, he has just previously stated to have "no shadow of variety or composition in it"; (*b*) the notion of a *collection*—his "collective idea"; (*c*) the notion of *serial order*; (*d*) the notion of *quantity*

[6] *Essay*, II, xvi, 5.

—greater and less; (*e*) the notion of a mathematical *operation*—addition, subtraction. Thus the main elements in the concept he is describing are assumed; at the same time there may be a seasoning of hard-headedness in his stout nominalism. For him numbers are names: "Without names or marks we can hardly make use of numbers in reckoning, especially where the combination is made up of any great multitude of units, which, put together without a name or mark to distinguish that precise collection, will hardly be kept from being a heap in confusion." One of the primary issues in the modern discussion of the nature of number is just whether supersensible (or superintuitible) mathematical ideas do not resolve into mere nomenclature and the science itself into a kind of transcendental logomachy.

That the Lockean type of nominalism is by no means extinct is evidenced by the definition of number offered in the *Encyclopaedia Britannica*:[7] "Suppose we fix on a certain sequence of names 'one,' 'two,' 'three,'. . . ., or symbols such as 1, 2, 3,. . . .; this sequence being always the same. If we take a set of concrete objects, and name them in succession 'one,' 'two,' 'three,'. . . . naming each once and once only, we shall not get beyond a certain name, e. g., 'six.' Then, in saying that the number of objects is six, what we mean is that the name of the last object named is six. We therefore only require a definite law for the formation of the successive names or symbols. The symbols 1, 2,. . . .9, 10,. . . ., for instance, are formed according to a definite law; and in giving 253 as the *number* of a set of objects we mean that if we attach to them the symbols 1, 2, 3,. . . . in succession, according to this law, the symbol attached to the last object will be 253. If we say that this act of attaching a symbol has been performed 253 times, then 253 is an *abstract* (or *pure*) *number*.

[7] 11th ed., article "Arithmetic."

Underlying this definition," continues the writer, "is a certain assumption, viz., that if we take the objects in a different order, the last symbol attached will still be 253. This, in an elementary treatment of the subject, must be regarded as axiomatic; but it is really a simple case of mathematical induction."

The presupposition of discontinuity and of serial order is as obvious in this last as in the two previously given accounts of the number concept. We set out with our known power of observing differences and naming things —perceptual discrimination and apperceptional unification; but by the time we have accomplished the office of Adam and are taking our earned rest, we discover that the names we have given are vicariously indifferent to the things of "first intention," and in addition that they have won for themselves a wholly novel and stringent interdependence, —the smoke of our experience has transformed itself into a hugely articulate Jinni, and, as by a miracle, number is manifest! Aristotle says,[8] "In general what exists in the essence of number, besides quantity, is quality; for the essence of each number is what it is when taken once, 6 being not what it is when taken twice or thrice, but what it is once, that is, 6." It is very apparent that a succession of qualitative discriminations will not in itself yield quantity; and without an understanding of quantity how can number be defined?

II.

The ideal of the logisticians (though I speak with misgivings) is at once the infallibility and the universal applicability of their reasonings. They would create for us a rational universe entirely freed from the taint of empiricism, mathematical in its certainties, but hypermathematical in its significance,—in short, they would

[8] *Metaph.*, 1020*b*.

achieve what Spinoza so greatly attempted. Because of the annoying miasmas which beset the earth-born speech of men, they would substitute therefor a kind of Esperanto of the soul (*anima intellectiva*) modeled after the discarnate and purified symbolism of mathematics. Clearly the approach to this consummacy of the intellect should be through the concept of number.

First of all, this concept must be relieved of all traces of Lockean empiricism. The simple notion, prevalent among the ordinary, that the idea of number is in some fashion derived from the act of counting is one of which we must be eased. For what is meant by counting? "To this question we usually get only some irrelevant psychological answer, as, that counting consists in successive acts of attention. In order to count 10, I suppose that ten acts of attention are required: certainly a most useful definition of the number 10!"[9] The point is well taken, and we can see that it applies conclusively to the whole British tradition, from Hobbes onward. "We must not, therefore, bring in counting where the definition of numbers is in question."

To be sure, this judgment has not prevailed in the new school *ab initio*. Dedekind states that from examination of what takes place in counting an aggregate of things, we are brought to consider the mind's powers (*a*) of relating things to things, and (*b*) of letting a thing correspond with, or represent, a thing; and that upon these powers as a foundation the whole science of number must be based.[10] *Relation* and *equivalence* are thus fundamental ideas—or, perhaps, operations—which get their meaning from counting, and give its meaning to number; but it may be that the counting here meant is of that purely noetic variety which includes "denumeration" of the in-

[9] Russell, *Principles of Mathematics*, p. 114.
[10] *Was sind und was sollen die Zahlen?*

finite along with "enumeration" of the finite, and which, putatively, owes no dependence to our commoner experience.

But if not counting, then neither is mathematical induction the key to the meaning of number; for mathematical induction, with its dual stress upon *next-to-next* and *recurrence*, is no more than the act of counting transubstantiated by that unity-in-variety which is the root of all perception. "We may define finite numbers as those that can be reached by mathematical induction, starting from 0 and increasing by 1 at each step,"[11] but such a definition does not apply to the vastly greater realm of transfinite numbers,—and it would be obvious waste to devote thought to a definition applicable only to the "little corner," as Poincaré calls it, "where the finite numbers hide themselves."

By what device, then, are we to pry into the mystery of number? What idea — which the mutations of the Wheel of Time have brought back to us freed from the contaminations of a too mortal birth—will give us its eluctant essence? The answer is familiar: A finite cardinal number is a class of equivalent classes; an infinite cardinal number is a class of classes a part of which is equivalent to the whole. It is the idea of *class* which is to resolve for us the riddle of reasoning.

Readily enough our imaginations seize the suggestion. The older, empirical conception of number as somehow directly derived from the act of counting, in reason as in history, is replaced by one in which counting and all other operations flow from an initial insight into a group situation. The point of regard has been reversed, and in place of seeing a perceptual situation built up out of moments, we see the moments emerge from the situation; logical priorism disenthrones empiricism, deduction precedes in-

[11] Russell, *op. cit.*, p. 123.

duction—and, indeed, not unnaturally absorbs the latter, for any induction which may lay a claim to reason is but deduction disguised.[12]

But this might flow from a mere distinction of temperament;[13] for we have long been accustomed to Urania and Pandemos in reason as in love. The matter which calls for a nicer determination is the relation of this term *class* to its content. What does it mean?

It is difficult to be precise in the analysis of terms which are customarily defined only by a set of properties couched in the form of postulates. What one arrives at is a word (*flatus vocis*) with a variety of meanings, but meanings eviscerated of that heart of reality which we feel to be present in our more current, if less critical, living speech. Indeed, all that saves this rarified discourse from the emptiness of nominalism is the requirement of consistency as between the postulates; their freedom from mutual contradiction is their sole claim to a single and central meaning. This (if I understand it) is the only principle of definition recognized in logistic.

What, then, are the properties of a "class"? Clearly, I think, the prime requisite is that it shall constitute a *limit.* I do not mean a limit which conveys a sense of a beyond (if that can be avoided), but a limit which clarifies our sense of the within,—such a limit as, for example, is represented by the cardinal number of the class of finite numbers, or again, such a limit as we ordinarily intend by the word "universe." Without this conscious limitation, which, because we feel it to be a voluntary intellectual retrenchment, a kind of rein upon the imagination, we personify

[12] *Ibid.*, pp. 11*n.*, 441.

[13] This, apparently, is Poincaré's notion of his own divergence from Russell. "M. Russell me dira," he says, "qu'il ne s'agit pas de psychologie, mais de logique et d'épistémologie; et moi, je serai conduit à répondre qu'il n'y a pas de logique et d'épistémologie indépendantes de la psychologie; et cette profession de foi clora probablement la discussion parce qu'elle mettra en évidence une irrémédiable divergence de vues."—*Dernières pensées*, p. 139.

as a "self-limitation," no conception of class could be operant.

Dedekind's solution of the problem of continuity quite consciously rests upon the assumption of *limits,* or limiting values; and what is distinctive of the notion of a "cut" (*Schnitt*) appears to be just that it determines a limit which, so to speak, does not overleap itself, and which consequently gives the base for a self-contained system of values. Every "cut" is, in a sense, a zero, having the particular property that any variable magnitude which approaches the limit loses itself in a value indistinguishable from zero.[14] This, I take it, is also the essential meaning of the *Nul class*—the class of things to which no entity in the (given) universe corresponds; it is essentially a boundary which, because it is empty, cannot be used as a turn or start into continued reasonings.

At least we should suppose that 0-limits could not be so used, but by a kind of transcendental induction just this is attempted. The cardinal number of all finite numbers, which is, of course, infinite, becomes the first transfinite cardinal; and the ordinal ω (ω symbolizes a *progression* modeled on the natural suite $1, 2, 3, \ldots n, n+1, \ldots,$ and so may be regarded as the generalization or law of the process of ordering sequentially) becomes the first transfinite ordinal. By applying the conception of a transfinite ordinal to transfinite cardinals, it becomes possible to conceive of, and perhaps create, an infinite series of the latter —transfinites of the order α being followed by those of the order β, and so on. The whole process is reminiscent of Spinoza's assumption of possible infinite attributes, other than thought and extension, of the divine substance, though it seems to want the restraint which left Spinoza content to suggest the possibility, and pass in his philosophizing to the attributive planes with which human experience famil-

[14] *Stetigkeit und irrationale Zahlen,* IV.

iarizes us. By means of such interplays of conception—infinite limiting finite, transfinite limiting infinite—it becomes possible to create whole hierarchies of classes and types, each conclusively including what is below it and conclusively ignoring what is above it. The process is interesting and in its way fruitful, but it is difficult to see how it could be possible except for that self-imposition of limits which distinguish grade from grade and type from type, and it is difficult to see in the imposition any other necessity than the arbitrary will of the thinker. The limits set are limits assumed, and assumed with something of the stark inexplicableness of a primitive tabu—unless we concede that the whole process is a conscious fiction, whose analogue is our empirical concentration of immediate attention on immediate ends.

But besides this external principle of limitation, which makes definable a self-comprehending system, there is another principle of limitation, an internal one, which makes system itself comprehensible. This principle is represented by the idea of *structure* or *form,* without which mathematics and reason alike could not exist. The principle of external limitation might suffice to mark off for us an islet of chaos which we could choose to regard as the universe, but only the acknowledgment of internal limitations could convert this chaotic universe into a cosmos.

Now the relationships of ideas according to this principle of internal limitation assume two general forms: that of *part-to-part* and that of *part-to-whole.* It is obvious[15] that each of these is a relation of *order,* and it is also obvious that each is derivative of the idea of *unity* in the two fundamental senses of unity. For the relation of part-to-whole clearly rests upon the contrasting unities of the element, regarded as an undifferentiated item, and the thing, regarded as an assemblage of elements; and the

[15] "Obvious," not to logistic, but to our linguistic intuitions.

relation of part-to-part, while explicitly concerned only with the relation of item to item, clearly rests upon an implicit whole.[16] Unit and totality, atom and universe, are the two extremes, each of which assumes the mask of unity, and the fact that the atom may be resolved into a universe or the universe contracted into an atom by a simple act of speculative translation does not alter the essential character of these two moments of thought.

The relation of part-to-part would, in the world experientially familiar to us, involve the meaning "next-to-next," or contiguity of consecutive elements. This relation is what makes the experiential world finite and incomplete; it is, therefore, felt as a constraint of the pure reason, mathematical or other. But the logisticians have discovered an escape from this restriction, and like Spinoza have found their freedom *sub specie aeternitatis*. The instrument of emancipation is the notion of the "one-to-one correspondence"; it is through this that the infinite is resolved

[16] For the two types of unity, cf. Bergson, *Données immédiates*, pp. 58f. Of course the logisticians categorically deny that the idea of *class* involves that of "part-to-whole." "Socrates is a man" may mean (1) "Socrates possesses the qualities which mark a human being"—and this is the part-to-whole relationship—or (2) "Socrates is one among men,"—and this is the member-to-class relationship, expressed "x is an a" (symbolically, $x \epsilon a$), where x is a member and a a class. The distinction is true enough, and it is also true that only the part-to-whole relationship is "transitive," i. e., subject to syllogistic treatment. But is it not evident that the distinction is fundamentally the very distinction which a philosophy of number is called upon to explain? Reasoning *qualitatively*, i. e., where your terms are taken "by nature," we get judgments of type 1; reasoning *quantitatively*, i. e., with terms taken "in respect to number," we get those of type 2. In judgments of the member-to-class type number is assumed, not definitely as if counted, but indefinitely as if countable. That is, a plurality, which is a totality or aggregate of some sort (in so far as limited by the reasoning undertaken), is at least hypothetically "taken"; and such a plurality is what is meant by a "class" (except in those shadowy extremes where the class has only one member or none at all). But if there is a plurality or aggregate it must have the configuration of just this (whatever it may turn out to be) aggregate which is being dealt with—just the class in question. Such configuration (which we might call the quality of a quantity) is precisely a whole of which the member is a part,—at least, we use "whole" and "part" in this sense in common speech, and it is certainly significant that the logisticians, in denying that "class" has this meaning, are forced to proclaim the term undefinable except by its use—i. e., it is left in a state of empirical ambiguity. Cf. Russell, *Principles*, Chaps. II, VI; also, Burali-Forti and Padoa in Vol. III of *Bibliothèque du Congrès international de Philosophie.*

into cosmos. The idea is centrally that of the reciprocal uniformity of two groups (classes), such that for every element of the one there is in the other, one, and only one, corresponding element. Two groups or classes so related are said to have the same number, and the infinite is simply a group in which the whole is related to a part of itself in this manner.[17]

Now the notion of a one-to-one correspondence is clearly metempirical. In real life, we cannot make things correspond absolutely except in absolute identification, i. e., in loss of plurality; all other relationships involve some kind of contiguity. Even when we set five fingers against five fingers, what we have empirically is not a one-to-one correspondence of two groups, but right thumb to left thumb, right index to left, and so on; and this holds throughout the empirical universe. The idea of number is, as it were, interposed between the severally adjacent digits, or perhaps I had better say that the groups of five are groups of five because they both (speaking with Plato) "partieipate" in τὰ μαθηματικά. The correspondence lies between an empirical group which is always finite and incomplete and a metempirical system of numbers (supermundane, if not divine) representing the class of all possible classes. If Spinoza's divine substance, within which all attributes inhere, were to become articulate it would be represented, I conceive, by just such transcendental numbers.

But we are not to think of these numbers as severally interdependent. Their reality rests upon no idea of succession. We must think of decads, duads, monads, triads, tetrads, etc., not of one....two....three....etc. The *order* of the numbers in their own transcendent realm is something superposed upon their cardinal realities,—this

[17] The usual illustration is that of the one-to-one correspondence of all the integers with all the even integers, or of the points on a straight line with the points on a plane.

time by a set of relations which concerns them *inter se*. *Less than, greater than, equal to,* or again *higher* and *lower power,* or again *betweenness* (or "mediacy," since the notion of "between" is significant only when coupled with the idea of transition), are relations of the needed kind. Now each of these sets of conceptions is a variant of the *part-to-whole* relation, of contained and container. This is self-evident in the first-named group. "Less than" and "greater than" obviously rest upon the experiment of mensuration, of reduction to scale, and if the numbers themselves *are* the scale, nevertheless they get their steps or intervals, and hence their *order,* from the experiences whose comparisons they name. A scale may be regarded as made up from the sucessive remainders in a series of approximations, its fineness being determined by the extent to which the approximations are carried,—which, in last resort, must be a matter of industry or of organic structure, in either case empirical. The relation of equality is not so obviously derived from measure, for "equal" may signify not merely identity in step or scale, but also similiformity and equivalence. Nevertheless, when we consider that equivalence is no more than functional identity and that similiformity can be no less than this—that is, that each of these ideas is identity with a reservation—it would seem evident that here too we are dealing with a concept whose final meaning is derived from the part-to-whole relation.

In the case of "higher-lower power" and in the case of "betweenness" the same general relation—part-to-whole—is implicit. Both of these types of expression are derivatives of space-perception; they are geometric in first intention. But as principles of order they have to do not with a static but with a dynamic geometry. The notion of direction or sense is the primary one, but the direction exists not as the expression of an orientation but of a

progression; not a set of starting-points or markers but a set of journeys is connoted. Thus we have time as well as space involved in the empirical foundation of numerical order so conceived, the complete idea being the analogue of a movement from any assumed position in any designated direction, the movement being conceived as contained by its determinants. Of course, in the case of "betweenness" this movement may be ideal, and in that case we have merely a case of syllogistic transition, with the "between" represented by the middle term;—but this is simply intellectualizing our journey. Again, the concept of "betweenness" may give rise to right-left, symmetrical-asymmetrical orders; but here, too, we have only special complications of the familiar idea, for right-left are clearly but alternative journeys, a dilemma of roads one or the other of which our action must make real (hence defining the whole);[18] while symmetry and its opposite can hardly be conceived apart from measurement, for indeed the whole notion of proportion is dependent upon some kind of repetition (which again throws us back upon time and space for our analogues).

Thus the logistic conception of number, starting with the assumption of *class* as the essential numerical idea, proceeds in two directions. (*a*) Outwardly, it posits a limit or law within which must fall all the elements which make the class a class, capable of structure. And that this outward limitation is made in good faith as essential to the idea is sufficiently evidenced by the recognized possibility of a class including classes, of a class of classes, and finally of the class of all possible classes,—a veritable hierarchy of types of limitation. (*b*) Inwardly, there are posited two types of structural relation, which may be described as the principles of internal limitation. These are the re-

[18] So also "before-after." Past time is commonly thought as a *retreat* from the present, future time as an *advance*.

lation of part to part and part to whole. From the first is derived that freedom to make comparisons which makes possible—or, *is* the possibility of—the transcendental independence that distinguishes pure number. From the second flows the whole concept of order, and especially the notion of series or progression without which the idea of quantity (i. e., greater-less) could not be.

If we ask what concepts are fundamental in such a construction, three seem to stand predominant: class, element, relation. But the two first, class and element, are surely no other than the two meanings which we commonly ascribe to unity, while relation is quite as clearly *the* function (and therefore the meaning) of plurality. The *one* and the *many* are thus the fundamentals of number,—and already we seem to be within hailing distance of the Hellenic categories; subject and attribute, thing and quality, are recurrently proximate. Has the Wheel of Time indeed completed its circuit? and is philosophy to begin anew? Or were we perhaps right as to the distinction of temperaments, and is logistic but an exercise of the lovers of Uranian reason?

III.

At the beginning of my discussion I quoted from Gaston Milhaud a word of caution in regard to that dogmatism which issues from a too naive confidence in the powers of our understanding, especially when freed, as it is in mathematical logic, to consume its own intentions. I would repeat this caution, having in mind certain developments of this logic based upon the principles already examined.

These developments issue from that abstractive freedom which is the especial pitfall of the Uranian mind. When in a given situation a given form is discovered, the statement of this form is what we call the description of the situation, for it is only forms that we can state. But

a form so abstracted—and this is the law of our rational life—is invariably made the measure of new situations. The fact that it can never be applied to a new situation except with some more or less accommodating deformation is a fact which we customarily and conveniently neglect, or if we remember it, it is only for the sake of abstracting from the more comprehensive situation given by the group of deformations a new form of forms which shall serve in its turn for the first of a series of modifications of some super-form of forms, and so on;—i. e., $sf \dots df \dots df' \dots df'' \dots$ etc., as it becomes clogged by the impertinencies of fact, is clarified by being transmuted into FS... DF... DF′ ... DF″... etc., and this in turn by εἶδος ... ἐντελέχεια $(\alpha' \dots \beta' \dots \gamma' \dots)$, whence, we may presume, the Idea of Ideas breathlessly emerges as we pass above the sphere which bounds our empyrean. If this description be false to the process it has not yet been so demonstrated.

Now there are two modes in which this process is applied in the logistic analysis of number, corresponding to the two types of relation of a class to its limits which we have heretofore stated. These two modes might be described as the modes of external and internal transcendence of unity.

The first of these, the external transcendence, is effected by analogical reasoning the base of which is the so-called "natural" suite of numbers, the succession of positive integers $1, 2, 3, \dots n, n+1, \dots$ The number of such integers, which is infinite, is ω; but ω is more than this. It is also *the principle of description which is immanent in the natural numbers naturally arranged*; it is the principle of numerical order as evinced in one-to-one correspondences, and so is the key to the analysis of all denumerable groups. The postulates underlying descriptions of the type ω are (a) the postulates of *linear order,* and (b) postulates of *sequence*—Dedekind's for example. From the

combination of these two ideas issues the conception of a discrete series, though when we consider that the first of these is symbolized merely by the idea of inequality ($<$, $>$), i. e., by quantity, and the second by that of *limit,* i. e., by class, it does not appear that "discrete series" spells much more than "whole numbers." Nevertheless, as symbolized in ω, it becomes the beginning of a transfinite hierarchy of orders; for it is the principle (or, shall I say, the analogy) of the suite of finite numbers which sets in order the houses of the infinite,—there the last becomes first, Omega the prior of Alpha, and the unity of the finite integers is transcended by numbers α_ν reaching to the order 2ω, while beyond this we may suspect yet more transcendent orders of hyper-α's.

But this external transcendence of unity is complemented by an internal transcendence; there is not only a metempirical macrocosmos, but a metempirical microcosmos. This is shown forth when in the description of order the notion of sequence is replaced by that of *betweenness* or mediacy, which is to be conceived as a kind of eternal negation of next-to-nextness without loss of plurality. There are two kinds of numerical order exemplifying this internal transcendency. When a series is endlessly linear and yet endlessly median, i. e., when it has no beginning nor middle nor end but only and always a median term between any two terms, it is *dense.* When a series is limitedly linear but has no middle term, it is *continuous.* The endless fractioning of a difference in a process of approximation—as, for example, the endless interstitial fractions required to complete the suite of all rational numbers—is image of the dense series; the clogging of an interval by the sum of its own possibilities is the image of the continuous series—for example, the series $\geqq 0 \ldots\ldots \leqq 1$ is fulfilled by the aggregate of numbers rational and irrational there comprised. Series of each of these types are trans-

finite; but there is an important difference in structure between them, for only the dense series is denumerable (i. e., figurable by the progression of positive integers), while only the continuum is susceptible of ratio and of measure, for it alone has limit. Of course the dense series is only metempirically countable and the continuous series measurable only metempirically, so that to note that we seem to have here naught but a transcendentalizing of the Aristotelian πλῆθος and συνεχής, plurality and magnitude, is to suggest an empirical meaning for what is by definition beyond experience.

And yet is this suggestion without reason? The transcendentalities of logistic are accomplished in two directions, which might be termed the gross anatomy and the histological analysis of the number-cosmos; and yet, in order that the directions may be meaningful, must we not recognize some proximate and experiential greater-less which is the *here* from which we orient these directions? This seems clearly implied by the important rôle played by the conception of the suite of "natural" numbers, and again by that of the line, in the representation of order. Very likely it is true that finite numbers cannot be satisfactorily defined except in relation to transfinite classes, but can the transfinite be defined without first assuming the finite? As a matter of fact, the transfinite orders seem all to be got by a process of progressive abstraction and recombination of qualities assumed on the analogy of the natural numbers; it is as if, by a cunning complexity of mirrors, the natural suite were made to suffer indefinite distortions, variously deforming its native properties and translating them from plane to plane and from space to space in a succession of *saltus,* as many as one has patience for.[19]

[19] This right of saltation is clearly the foundation of the conception of transfinity. "In recent times there is arisen, in geometry and in particular in the theory of functions, a new type of conception of the infinite; according

The process is legitimate enough if we be not duped by its parlous illusions. That is, we must preserve our sanity (which is nothing less than our common-sense faith in our common-sense intuitions); and for this I can conceive no better rules than are implied in Aristotle's dicta (1) that when we speak with reason we must say something with a communicable meaning, and (2) that "third man" abstractions are wasted breaths.[20] The first of these is a pragmatic statement of the law of contradiction applied to discourse; the second is a caution against the tautology involved in the regress to infinity. If we adhere to the first we cannot shift our perspective (say, from finite to transfinite) without distortion of meaning, i. e., without altering our predication; if we adhere to the second we cannot make abstractions of abstractions without losing reality altogether.

Now it would seem that logistic fearlessly invites both of these perils. In the description of classes, for example, the forms of expression travesty the sense of language. For what can be the common-sense, linguistic meaning of a *Nul-class,* which must be described as that class which contains no element, or of which the universe (of discourse) furnishes no instance? Or again, by what right

to these new notions, in the study of an analytic function of a complex variable magnitude usage calls for the representation, in the plane which represents the complex variable, of a unique point situated in the infinite, that is to say, infinitely distant, but nevertheless determined, and for the examination of the manner in which this function comports itself in the neighborhood of this absolute point as in the neighborhood of any point whatever. It is seen then that the function in the neighborhood of the point infinitely remote acts precisely as it would act in that of every other point located in the finite, so that one is fully authorized in this case to represent the infinite as transported to a point altogether determined. When the infinite is presented in a form thus determined, I call it *infinite properly so-called.*"—G. Cantor, *Acta Mathematica,* 2, p. 382. Poincaré founds his conception of dimension upon the notion of the "cut" (*Dernières pensées,* p. 65; *La valeur de la science,* 97f), which, since it implies a new law in each new location, seems a more legitimate use of the right (or intuitive power, as Poincaré would make it) of overleaping boundaries ideally set. Plato's conception of the cosmos as made up of intervals and limits held together by proportion is not far from this (Timaeus, 35-36, 53-57; cf. Philebus, 14*c*-27*b*).

[20] *Metaph.,* 1006*a*.

of speech may we speak of the class *a* as the "class" which contains only *a*? In the first of these cases we are using the language of plurality about nothing, and in the second about one. And if we go a step further and speak of x' as the sole element of the class whose sole member is x, is this more than a vicious play upon the conception of part and whole?[21] Beyond this there is the x'' which is the sole content of the class whose sole element is x',—and we are fairly launched in the infinite tautologies of the "third man."

Formal refinements of analysis, when freed from the leadings of empirical need, may defeat the very end of analysis. Thought becomes not purified, but anemic; in a world of ideas where not only is language replaced by symbols, but these by symbols of symbols, all linguistically ineffable, it is small wonder that identities and the sense for the law of identity vanish away, so that no longer, in order to reason, do we need to speak significantly as Aristotle would require, nor indeed to speak at all. And with the disappearance of identities from this analytic attrition, it is but to be expected that there will emerge that "liberty of contradiction"[22] which solves infinity by denying sense and confounds truth with paradox. The ultimate reason of the world becomes a relation of relations which, if it could say anything, would say just that the world exists, catholically comprehensive of all contradictions, but which, since it is unutterable, is in so far inferior to the sacred monosyllable *Om*.

I am quite willing to agree that there is a sense in which this world is the best possible world, and indeed a sense in which it is the only possible world, and even a sense in which it is all possible worlds,—but when I have got so far I begin to suspect that I am being duped by my

[21] Perhaps I should mention that Burali-Forti *et al.* make distinction between "is an element of" and "is contained by."

[22] Cf. Poincaré, *Science et méthode*, pp. 195f.

own tongue and I deem it the modesty of reason to conserve my breath. Is there no like nonsensicality in the refutation of the axiom that the whole is greater than the part? And have we made "infinity" a more usable notion (I will not say in logistic, but in reason)[23] because we can juggle a part into a kind of equality with its whole by nominalizing our definitions? Common sense, we may be sure, will be slow to relinquish the intuitions upon which it acts; and will not our humaner reason itself, when it meets contradiction assuming the guise of infallible truth, begin to suspect that the ghosts of Duns and Occam are coruscating behind the scenes?

IV.

The attempts of the logisticians to define number by the unaided agilities of the reason are, in the end, little more satisfying than is the confident empiricism of Locke. No one can question their demonstrations, granted their premises; but no one, in the right mind of common sense, can grant the premises. It is incumbent upon us, then, to ask whether logistic has, after all, quite so efficiently scotched the particular theory whose downfall it proclaims,—I mean the intuitionism especially associated with the name of Kant.

"The pure form of all quantities for the outer sense,"

[23] I seem to discern among the logisticians themselves, when they are speaking the language of philosophy, a tendency to employ the idea of what Cantor terms "the infinite improperly so called" in place of that "properly-called" logistic infinite which could hardly be expected to convey an intelligible idea when severed from its nominalistic illustrations. And speaking of these illustrations, why should we stop with an infinite whose part equals its whole? Suppose we define Chaos as a nul-class ($X = 0$), and Cosmos as the class of all ordered classes, infinite in number ($K = \omega$). Then τὸ ὅλον, the Whole, (H), will be equal to a part of itself, a logistic infinity ($X + K = H$, or, $0 + \omega = \omega$). But suppose, in addition to infinite K, the Demiurge (since that is his business) determine other ordered classes (K'), as many as he may choose. K' will belong to K, as being ordered classes, but cannot add to the number of K which is infinite, nor to H with which K is already in a complete one-to-one correspondence. Then $K (= K + K') > H (= K + X)$, and the part is greater than the whole. Of course this is a play upon the idea of progression in time; perhaps none the less a fair image of the course of reason,—though "I hold it not honesty to have it thus set down."

says Kant,[24] "is space; the pure form of objects of sense in general is time. But the pure *schema of quantity,* as a concept of the understanding, is *number,* which is a representation conceptually combining the successive addition of unit to like unit. Thus number is nothing other than the unity of the synthesis of the manifold of a homogeneous intuition in general, in that time itself is engendered in the apprehension of the intuition."

Thus for Kant "unity in the apprehension of a manifold" and "time," the empirical image of an *a priori* schema, are the fundamentals of the idea of number. We shall be not far wrong in identifying here the notions of unity, multiplicity, and serial order, which are primitive with Locke and are unevaded by the logisticians. But Kant puts these notions in a somewhat new light: they are no longer *bloss empirisch,* as with Locke, nor are they circuitously inferred from nominalistic definitions; rather, they come into being as elements of that synthetic activity which is the dominant mark of mind. Number is, in this sense, neither empirical nor quite metempirical. The categories of the understanding lie behind the numerical schema, but the schema itself is "only the phenomenon or sensible concept of an object in agreement with the category." Further, this schema—as indeed are schemata in general—is only the *a priori* determination of temporal intuitions, getting its content not through analytic but through esthetic transcendentalities. Indeed, one is tempted to say that Kant, like Plato, puts his mathematical realities in a kind of mid-realm participating at once in νοῦς and αἴσθησις.

The unique position of the number idea appears again in Kant's discussion of the formation of determinate numbers. Judgments of numerical relations, he says, are certainly *a priori* syntheses, but they are not, like the under-

[24] *Kritik der reinen Vernunft,* 182.

lying principles of geometry, universal in character. Accordingly, they are to be termed *number-formulas* (*Zahlformeln*), not axioms, and they are endless in number, i. e., as many as numbers themselves.[25] Kant conceives the formative judgments as synthetic apprehensions of aggregations of units. In their generation we may make use of sensible intuitions, as in computing by aid of the fingers, but the actual realization of a sum would be impossible apart from the *a priori* schema. "The arithmetical judgment is always synthetic, as may the better appear when we consider the larger numbers; for it is then clearly evident that, apply our concepts as we will, without the help of intuition, by mere conceptual division into elements, we can never discover a sum."[26]

Couturat retorts upon Kant that it is practically impossible to have precise and complete intuitions of numbers of the order of millions, and that these could never be calculated exactly if recourse to intuition were necessary. "What is true of the large numbers," he continues, "is true also of the small, and consequently it is not intuition but reason that enables us to say that 2 and 2 make 4."[27] Evidently Couturat overlooks the case of the phenomenal calculator who handles millions as the average mortal handles units, and without being able to analyze the process; or again, the undoubted fact that the average civilized man would be a mathematical prodigy to the average primitive. And again, it is not easy to see that there is a more excessive dogmatism in assuming that our intuitions of the great numbers are in character with our intuitions of the small, than in asserting that because we have no intuitions of the great (supposing this true) we can therefore have none of the small,—which is Couturat's position.

Nevertheless, there is justice in Couturat's criticisms,

[25] *K. d. r. V.*, 205-6.
[26] *Ibid.*, 15-16.
[27] L. Couturat, *Les principes des mathématiques*, p. 256.

especially to the effect that Kant's notion of "numerical formulas," calling as it does for an infinity of irreducible synthetic insights, ill conforms to our notion of rationality, and is, indeed, only a masked intrusion of the old empirical view of number. The difficulty with Kant's view is that the number syntheses reduce to no law, which offends our sense of the reasonable, hyper-conscious as it is when touched on the side of mathematics. Kant's *a priori* synthesis is after all only a designation, and, as Poincaré says, to christen a difficulty is not to solve it.

Poincaré's own view—which may be described as Kantian with a saving salt of empiricism—is an interesting variation. The foundation of the idea of number is mathematical induction, and the essence of mathematical induction is reasoning by recurrence, while reasoning by recurrence has for its proper character just that "it contains, as it were condensed into a single formula, an infinity of syllogisms." Such a rule cannot come to us from experience; experience can show it to hold for a limited portion, but only for a limited portion, of the endless suite of numbers. If it were a matter only of this limited portion the principle of contradiction would suffice, permitting us to develop as many syllogisms as we wish; but when it comes to embracing an infinity in a single formula, when the infinite is in question, then this principle fails, and it is just here too that experience is impotent. The rule of recurrence, "inaccessible alike to analytic demonstration and to experience, is the veritable type of the synthetic judgment *a priori*." [28]

Why, then, Poincaré asks, does such a form of judgment impose itself upon us so irresistibly? "Because it

[28] *La science et l'hypothèse,* Chap. I. Cf. p. 37: "Nous avons la faculté de concevoir qu'une unité peut être ajoutée à une collection d'unités; c'est grâce à l'expérience que nous avons l'occasion d'exercer cette faculté et que nous en prenons conscience: mais, dès ce moment, nous sentons que notre pouvoir n'a pas de limite et que nous pourrions compter indéfiniment, quoique nous n'ayons jamais eu à compter qu'un numbre d'objets."

is only the affirmation of the power of the mind which knows itself capable of conceiving the indefinite repetition of an act once this act is found possible. The mind has a direct intuition of this power; experience can be only an occasion for making use of it and hence of becoming conscious of it."

But there is another and an important feature of reasoning by recurrence which Poincaré emphasizes, and this is the *inventive* character of its judgments. They are not only intuitive, born of the nature of the mind, they are also creative; and indeed it is mathematical induction alone which can apprise us of the new. Each number, then, is to be looked upon as an invention—not due to physical experience, but a self-discovery of the mind. But invention and the self-discovery of the mind do not cease so long as life lasts; and so, says Poincaré in another connection,[29] "when I speak of all the whole numbers, I mean by that all the whole numbers that have been discovered and will one day be discovered.... And it is just this *possibility* of discovery that is the infinite."

The psychological temper of this view is apparent; in so far it is empirical. But the validity of mathematical judgments is independent of the vagaries of experience; it is derived from the structure of the mind rather than from the accidents of a conscious life, and in so far the judgments are *a priori* and metempirical. Whether mathematical truths represent not only the organization of mind but also the organization of nature is an epistemological question for which Poincaré suggests an interesting answer, but it is properly a question, not of mathematics, but of metaphysics.

Analogous to Poincaré's view is that of Bergson, which also must be regarded as Kantian in type. Bergson begins his analysis of the number concept with the categories of

[29] *Dernières pensées*, p. 131.

unity and multiplicity: every individual number is to be regarded as a ratio between the one and the many, unit and totality. "There are two species of unity," writes Bergson,[30] "the one definitive, which will form a number in adding itself to itself; the other provisional, that of this number which, in itself multiple, borrows its unity from the simple act by which the intelligence perceives it. And it is undeniable that when we image to ourselves the unitary components of the number we believe ourselves to be thinking of indivisibles, this belief entering as a considerable factor in the notion that we can conceive number apart from space. In every case, viewing the matter more nearly, we shall see that each unity is that of a simple act of the mind, and that, this act consisting in uniting, it is necessary that some multiplicity serve as its matter."

The two poles of the idea of number, unity and multiplicity, correspond in Bergson's view to the subjective and objective elements of experience,—ultimately and respectively to time and space, use and generation. "You can never draw from an idea which you have constructed more than you have put into it, and if the unity with which you compose your number is the unity of an act and not of an object, no effort of analysis can evoke from it more than unity pure and simple. Without doubt when you equate the number 3 to the sum of $1 + 1 + 1$, nothing prevents you from holding as indivisible the units which compose it, but this is because you do not utilize the multiplicity with which each of these units is big. It is, moreover, probable that the number 3 presents itself to our mind in this simple form, because we are thinking rather of the manner in which we obtained it than of the use we can make of it. But we ought to see that if all multiplication implies the possibility of treating any number soever as a provisional unity which will add itself to itself, in-

[30] *Les données immédiates de la conscience*, pp. 58-65.

versely the units in their turn are veritable numbers as great as one may wish, though one provisionally assumes them to be indecomposable in order to combine them *inter se*. Moreover, by the very fact that the possibility of dividing unity into as many parts as are desired is admitted it is regarded as extended." In fine: "What properly pertains to the mind is the indivisible process by which it fixes its attention successively upon the diverse parts of a given space; but the parts thus isolated are conserved in order to be added to others, and once added among themselves they are open to a new decomposition of whatever sort. They are then parts of space, and space is the matter with which the mind constructs number, the milieu in which the mind places it."

Thus in the Bergsonian view numbers are ratios mediating time and space. The order in which they fall is first of all the order in which achieved experience presents itself, i .e., it is spatial. But space-perceptions are all provisional in character; consequently numbers are all provisional in character. Numerical order is not continuous, but composed *per saltum* (*par sauts brusques*); we form our numbers turn by turn, each assuming the character of a mathematical point separated by an interval of space from the point following, but as we recede in our series from the points first formed these tend to unite into a line, their synthesis being the necessary consequence of our averted attention. But "once formed according to a determinate law, the number is decomposable according to any law whatever"; and here we reach the apparent freedom and apriority of the mathematical reason, a number in course of formation is not the same as a number once formed; it is only the latter that is really divisible.

Doubtless to minds enamored of the eternal, Bergson's view will seem a veritable anarchy; perhaps metaphysically it is so; but it can hardly be denied that it gives a

fair description of the manner in which we actually learn and apply our numbers, and it gives also an intelligibility to the old-fashioned notion that number is generated by successive acts of attention which the old-fashioned explanations do not possess. This is due, of course, to the assumption of an intuitive reason, differing from Kant's—as with Poincaré—chiefly in its more direct reliance upon the course of conscious events, upon psychology conceived as mental history.

Nor is it altogether fanciful to see in Bergson's view a striking analogue of Plato's. Like Plato he conceives number as essentially a ratio. Like Plato he conceives the realm of numbers as a median realm, uniting the one and the many, participating in the one direction in the essential unity of thought, in the other expressing itself as the multiplicity of things. Number is the category which unites subjective and objective, ideal and material,—or in Bergsonian terms, time and space.

V.

The types of definition of number which we have been considering raise certain inevitable issues—none more inevitable than the question of the relation of psychology to logic, and of both these sciences to epistemology.

If we contrast the older empirical conception of number with the logistic view, we see at once that the former defines number from the point of departure of number genesis while the latter analyzes its nature irrespective of its origins. From this we may guess both the reason for the dependence of the older conception upon the act of counting, in the definition of number, and the reason for the aversion to counting (for their denials of its significance amounts to this) on the part of the logisticians. For there can be no question that, historically considered, the invention of counting is the beginning of a science of number;

nor again, that a study of the number-systems of primitive peoples, and indeed of the civilized, yield a direct insight into the modes in which numbers are thought. The psychology of number-consciousness is, therefore, a direct key to our mathematical use of numbers.

But is there another and more efficient key, not perhaps explaining the nature of our consciousness of numbers, but explaining why they are found to be applicable to experience or even susceptible of metempirical developments? To this question the logisticians respond with a various affirmative, "various" because, while for some logistic is a purely nominalistic science (or, more correctly, purely symbolistic), for others it is the clue to a realism transcending the fictions which impair all empirically-originated speech.

It must be owned that there is a kind of experiential warrant for each of these views—the Uranian as well as the Pandemian. For if the latter can appeal to the universal conformity of number notions in process of formation, to our physical and mental structure and needs, the Uranian reason can retort with the universal and seemingly superhuman validity of mathematics. Mathematical demonstrations need only to be understood in order to be convincing, and if there be such a thing as infallibility there can be no test for it save this. From such infallibility the Uranian may infer, with a show of force, that number is not the product of our experience, but is imposed on us by the structure of the universe. Mathematical truth is, at all events, more universal than anything else we know.

But this is a doctrine, not of logic, but of epistemology. Uranians do not like the word,—it has psychological associations. They prefer to mark their own science as at once hyper-psychological, hyper-epistemological, hyperlogical—a science which can have no name, since every name is contaminated with the experiential (humanly ex-

periential) references of language. They aim rather at a system of symbols which shall be untalkable, though catholic of the meanings of speech as well as of all other meanings;[31] they would introduce us into a sphere where human relations and merely human thinking are merged into the crystalline structure of the de-reified reality of a cosmos transcending speech.

"Extravagant realism" is the only historic caption that can fit this point of view, and extravagant realism is the philosophical creed which Russell at least is ready to make his own.[32] That some adherents of the movement balk at this is no matter of surprise; but surely it is with ill reason, for the philosophic alternative which is left them is a nominalism without even the consolations of speech. When symbols are refined to such an extent that they are but the symbols of systems of unutterable ideas, whose generality outgeneralizes nature, then surely their inventors are worse than dumb; they have become cousin-german to the apostles of the flux, and, with Cratylus, nothing is left them but to wag impotent digits.

When the rigorous following out of the mathematical reason leads to such extreme views, we may well bear in mind M. Milhaud's caution against a too naive confidence in the dogmatisms of our understanding. We may well ask by what right (since it is from no definable experience) transcendental realism justifies its *ex cathedra* affirmations; or, with Poincaré, what value is to be attached to a symbolism so ineffable that no testimony of familiar fact can sustain it. And we will surely be led to inquire if there be not some secure middle way, satisfying at once to our reason and our sense.

[31] I can imagine no more downright statement of the point of view than that of A. Padoa (*Bibliothèque du Congrès International de Philosophie*, 1901, Vol. III, pp. 317f.). Surely, when we are told that science is the peril of logic, that reasoning in order to be safe must be empty, we may well draw heretical breaths!

[32] *Monist*, October, 1914.

Now it would be presumption to affirm that the Kantian view—which we might term the "moderate realism" of the development—even as amended by Poincaré and Bergson is wholly satisfying. There are unquietable difficulties besetting every relativism, and these become accentuated when the relativity is between such extreme factors as reason and sensibility. It is far more comfortable to fashion a shapely abode of ideas of a single order and name it intellect than to be faithful to all the factors that enter into the cognizable world; nevertheless, it is only with this inclusive faithfulness at once to fact and to reason that temperaments of a certain kind can find their rest.

Herein is the merit of the neo-Kantian view. It sees the crudities of the old naive empiricism quite as clearly as do the logisticians; but for all that it is unwilling to abandon empirical leadings or to deny the centrality of our human experience, for mathematical as for all other meanings. Indeed, it asks, and asks fairly, of the logisticians by what right they assume that the numbers and measures that tell and mete the physical world are only illustrative cases to be subsumed under some cosmic Number, super-human and supra-mundane. Why, for example, is "the suite of *natural* numbers" so named, and why made the model for the conceptualization of all other series, if it be not due to some greater intimacy of nature which number has with this suite than with the others?

Referring to the arithmetical definition of continuity Poincaré says:[33] "This definition makes a ready disposal of the intuitive origin of the notion of continuity, and of all the riches which this notion conceals. It returns to the type of those definitions—so frequent in mathematics since the tendency to arithmetize this science—definitions mathematically sound, but philosophically unsatisfying. They replace the objects to be defined and the intuitive notion

[33] *Dernières pensées*, p. 65.

of this object by a construction made of simpler materials; one sees indeed that one can effectively make this construction with these materials, but one sees also that one can make many others. What is not to be seen is the deeper reason why one assembles these materials in just this, and not in another fashion." And again:[34] "Among all the constructions that one can make with the materials furnished by logic, a choice must be made; the true geometer makes this choice judiciously because he is guided by a sure instinct, or by some vague consciousness of I know not what geometry more profound and more hidden, which alone makes the value of the edifice built."

There is a sense, as we have said, in which the world *is* all possible worlds; but there is a commoner and more valuable sense according to which the world we call real is only one among many possible worlds. The problem at once of philosophy and of all rational life is to tell us just what this unique reality is, why the materials of creation have been assembled in just this, and not in another fashion.

Both Poincaré and Bergson recognize in mathematical reasoning a power or enterprise of the spirit which is in some sense prior to experience. It is in this that they are Kantians. This power, or intuition as they agree in calling it, gives to mathematical truths their sanctioning validity. But the validity of mathematics is not supposed, as with the logisticians, to derive from a firmament above the firmament; it holds only within the ranges of human insight, and indeed it is the definition of the utmost reach of this insight. "When I speak of all the whole numbers, I mean by that all the whole numbers that have been discovered and will one day be discovered....And it is just this possibility of discovery that is the infinite," says Poincaré. If I read Bergson aright, I judge his conception of

[34] *Science et méthode,* p. 158.

the unity of living time, within which number is generated in the perception of differences, to be not radically divergent from Poincaré's meaning; and certainly their common view squares with the kind of interpretation which language can give of number, and which the ordinarily thoughtful intelligence can accept.

Nor do I hesitate to add that its metaphysical implications are rich and profound. For a view of number which, while holding it within the leash of human experience makes of it the measure of our expectation of life, is surely sufficiently grandiose for any imagination, if it seem to make that expectation infinite. The intuition which gives the sanction becomes the testimony to a truth in number transcending the facts to which it is applied—that is, the little range of life here present—though not transcending the possibilities of real experience. Plato found in mathematical intuitions recollections from a previous life of the intelligence, Bergson and Poincaré treat them rather as prophecies of life to come; but these are only variations of a common doctrine.

Hartley Burr Alexander.
University of Nebraska.